Rutgers University: Algebra Written Qualifying Exam

August 2017: Problem 3 Solution

Exercise. List, up to isomorphism, all finite abelian groups G such that the order of every element of G divides 55, and the number n_{55} of elements of order exactly 55 satisfies

$$10^2 \leq n_{55} \leq 10^3.$$

You must prove that your list is accurate.

Solution.

G can be written as direct sums of $\mathbb{Z}_5$ and $\mathbb{Z}_{11}$.

If $\mathbb{Z}_{p^k}$ is in the direct sum G will have an element of order $p^k \nmid 55$.

$\mathbb{Z}_5 \oplus \mathbb{Z}_{11}$ is obviously too small.

$\mathbb{Z}_5 \oplus \mathbb{Z}_5 \oplus \mathbb{Z}_{11}$:

(a, b, c) has order 55 IFF either a or b is nonzero AND c is nonzero.

$$n_{55} = (4)(5)(10) + (1)(4)(10) = 240 \qquad \checkmark$$

$\mathbb{Z}_5 \oplus \mathbb{Z}_{11} \oplus \mathbb{Z}_{11}$:

Let $(a, b, c) \in \mathbb{Z}_5 \oplus \mathbb{Z}_5 \oplus \mathbb{Z}_{11}$ s.t. $a \neq 0$ $\underline{\text{and}}$ $b \neq 0$ $\underline{\text{or}}$ $c \neq 0$

$$n_{55} = (4)(10)(11) + (4)(1)(10) = 480 \qquad \checkmark$$

$\mathbb{Z}_5 \oplus \mathbb{Z}_5 \oplus \mathbb{Z}_5 \oplus \mathbb{Z}_{11}$: (note: this is the next smallest direct sum of $\mathbb{Z}_5$ and Z_{11})

Let $(a, b, c, d) \in \mathbb{Z}_5 \oplus \mathbb{Z}_5 \oplus \mathbb{Z}_5 \oplus \mathbb{Z}_{11}$ such that $a \neq 0$ or $b \neq 0$ or $c \neq 0$ and $d \neq 0$

$$(4)(5)(5)(1) + (1)(4)(5)(10) + (1)(1)(4)(1) > 1000 \qquad \text{X}$$

$\implies$ All remaining direct sums of $\mathbb{Z}_5$ and $\mathbb{Z}_{11}$ will be too large.

$\boxed{\mathbb{Z}_5 \oplus \mathbb{Z}_5 \oplus \mathbb{Z}_{11}}$ and $\boxed{\mathbb{Z}_5 \oplus \mathbb{Z}_{11} \oplus \mathbb{Z}_{11}}$